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FLUID FLOW AND HEAT TRANSFER IN THREE-DIMENSIONAL DUCT FLOWS

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Abstract—A calculation procedure is described for three-dimensional duct-flow situations which are partially-parabolic in nature, i.e. those in which convective influences pass only downstream, diffusive influences are directed across the stream, but influences are transmitted from downstream regions to upstream ones by way of pressure. The numerical calculation procedure handles such flows economically; it stores the pressure as a three-dimensional array, but other variables two-dimensionally. As an illustration, the results from an application of the calculation procedure are compared with those of a parabolic calculation procedure.

NOMENCLATURE

A, B, C, D, coefficients in the finite-difference equations;
J, diffusion flux;
K, constant;
p, pressure;
S, source or sink term;
u, velocity along x-direction;
v, velocity along y-direction;
w, velocity along z-direction;
x, y, z, coordinate directions.

Greek symbols

ρ, density;
τ, shear stress;
ϕ, general variable.

Subscripts

D, E, W, N, S, U, P, e, w, n, s, p, refer to grid and interface locations;
x, y, z, coordinate directions;
u, v, w, refer to corresponding velocities.

1. INTRODUCTION

1.1. *Classification of steady-flow situations*

IT HAS been useful in numerical fluid mechanics to divide steady-flow problems into two classes: elliptic and parabolic. Strictly speaking *all* flows except wholly supersonic ones are elliptic; this means that perturbations of conditions at any point of the flow can influence conditions at any other point. The mechanisms of these interactions are usually:

(i) Convection (i.e. downstream transmission along stream lines);
(ii) Conduction, diffusion and viscous action (i.e. dissemination in all directions by molecular intermixing);
(iii) Pressure transmission (e.g. the tendency of a fluid in a subsonic flow to move out of the way of a downstream obstacle before reacting it).

In "parabolic" flows, mechanisms (ii) and (iii) are weak enough to be ignored; and the flow configuration is free from "recirculation", so that mechanism (i) transmits effects only in one direction. Many boundary-layer, duct-flow and jet phenomena are of this parabolic kind; for often the Reynolds number is high enough to render the molecular actions insignificant in the streamwise direction; and the boundaries of the flow domain provoke no sharp curvatures of streamlines.

In the present paper however, attention is focussed upon a class of flow situations which is intermediate to the parabolic and elliptic categories. Such flows, here called "partially-parabolic", are characterised by:

(a) Absence of recirculation, so that mechanism (i) (convection) operates only in a single (downstream) direction;
(b) High Reynolds number, so that mechanism (ii) (molecular action) is significant only normal to the stream-lines;
(c) *Significant* curvature of boundaries, rendering (iii) (pressure transmission) the dominant transmitter of influences in an upstream direction.

1.2. *Examples of partially-parabolic flows*

Phenomena falling into the partially-parabolic class include:

(a) Flow in strongly-curved ducts, for example pipe bends in heat exchangers;
(b) Flow in turbine and compressor cascades;
(c) Flows in and near partially permeable resistances such as gauzes and screens of tubes or rods, as in the shells of some steam generators;
(d) Flows of lubricants in two-dimensional oil films.

1.3. *Significance for numerical computation*

Elliptic flows require computer storage of dimensionality equal to that of the flow; the storage dimensionality of a parabolic flow, by contrast, is one less than that of the flow. Consequently, since influences spread only in the downstream direction in parabolic flows, *marching integration* can be employed; and there is no need to retain in store flow variables for more than the immediately-upstream plane or line. For elliptic phenomena by contrast, it is necessary to retain all upstream values in store; for they may have to be altered again in the light of adjustments still to be made downstream: an iterative procedure is thus always required.

For *partially*-parabolic flows, the requirements are intermediate: only the pressure requires to have storage dimensionality equal to the flow dimensionality; the other variables (i.e. velocity components, temperature, concentrations, etc.) require only the reduced dimensionality of parabolic flows. Thus the main advantage of a partially-parabolic situation, over the elliptic one, comes from the significant reduction in the storage requirement. This advantage is greatest for three-dimensional flows, as can be seen in the following calculations:

Suppose that 20 grid points are required in every direction for adequate coverage of the domain, i.e. 400 for a two-dimensional problem and 8000 for a three-dimensional one.

Suppose also that we are concerned, as is often the case, with three velocity components, pressure, temperature, two turbulence quantities and concentration, i.e. eight variables in all. If the flow is two-dimensional and elliptic, we need 8×400, i.e. 3200 storage locations; however, if it is partially-parabolic the storage is reduced to 400 (for pressure) $+ 7 \times 20$, i.e. to 540 locations, a reduction of 2660.

A three-dimensional elliptic problem with this grid fineness requires 64 000 storage locations; if however the process reduces to partially-parabolic form, the storage requirement is only $8000 + 7 \times 400$, i.e. 10 800, a reduction of 53 200. Such a reduction is of great value.

This being the case, it is perhaps remarkable that the partially-parabolic flow class seems to have escaped attention until now. Certainly there is every reason to recommend that wherever possible, three-dimensional flows should be treated as partially-parabolic instead of fully elliptic.

1.4. *Outline of the present contribution*

Calculation procedures for three-dimensional parabolic [2] and elliptic [1] flows have been available for some time; and they have been applied to various flow configurations. In this report we describe a numerical procedure for the calculation of *partially-parabolic* flow situations. Like the parabolic calculation procedure [2], the present procedure is of a finite-difference type and makes use of the SIMPLE* algorithm; but its distinctive features are:

(a) The pressure field alone is stored in a three-dimensional array, to be used in common for all the three momentum equations.

(b) An iterative, marching-integration procedure is adopted, whereby several sweeps of the flow domain are made; each sweep uses a better estimate of the pressure field, deduced from the observation of errors during the previous sweep. All other variables, e.g. velocities etc., are stored in two-dimensional arrays.

In Sections 2 and 3 of the report, the differential equations and the calculation procedure are explained; an illustrative example of partially-parabolic flow situation is described in Section 4, along with the application of the present procedure for its calculation. From comparisons made between the results using the parabolic and partially-parabolic procedures, it is observed that the partially-parabolic calculations display the expected flow-pattern and differ significantly from those obtained by using the parabolic procedure.

2. DIFFERENTIAL EQUATIONS SOLVED

The equations governing a partially-parabolic flow are the familiar Navier–Stokes equations for a steady flow but with diffusion in the predominant flow direction (z) neglected. In the (x, y, z) coordinate system, they are:

Mass conservation:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \qquad (2.1)$$

x-direction momentum:

$$\frac{\partial}{\partial x}(\rho uu) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho wu) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + S_u \qquad (2.2)$$

y-direction momentum:

$$\frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho vv) + \frac{\partial}{\partial z}(\rho wv) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + S_v \qquad (2.3)$$

z-direction momentum:

$$\frac{\partial}{\partial x}(\rho uw) + \frac{\partial}{\partial y}(\rho vw) + \frac{\partial}{\partial z}(\rho ww) = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + S_w \qquad (2.4)$$

Transport of a scalar property, ϕ:

$$\frac{\partial}{\partial x}(\rho u\phi) + \frac{\partial}{\partial y}(\rho v\phi) + \frac{\partial}{\partial z}(\rho w\phi) = \frac{\partial J_\phi}{\partial x} + \frac{\partial J_\phi}{\partial y} + S_\phi. \qquad (2.5)$$

In the above equations, u, v and w denote the velocities along the x, y and z directions; ρ represents the fluid density, and p the pressure. The "τ"s represent the shear stresses in the fluid; and J_ϕ stands for the flux of the property ϕ. The terms S_u, S_v, S_w and S_ϕ represent additional sources or sinks.

The differences between the above equations and those of elliptic and parabolic flows are the following:

(i) For an elliptic flow, the governing equations will contain also the shear stresses in the z direction; i.e.

*SIMPLE stands for semi-implicit method for pressure-linked equations.

terms such as $\partial\tau_{xz}/\partial z$, $\partial\tau_{yz}/\partial z$ etc. will appear in the corresponding equations.

(ii) For a parabolic flow, on the other hand, not only do the equations *not* contain the diffusion fluxes in the z direction but separate pressure fields have to be employed for the lateral and longitudinal momentum equations. The latter practice in parabolic flows is necessary to ensure that the pressure transmission of downstream events is negligible.

3. DETAILS OF THE SOLUTION PROCEDURE

The above-described differential equations for a partially-parabolic flow are solved using a finite-difference calculation procedure. The calculation procedure is based on the numerical algorithm called SIMPLE (for Semi Implicit Method for Pressure-Linked Equations) which was developed earlier by Patankar and Spalding [2] for parabolic flows. Because of the similarity in the equations for parabolic and partially-parabolic flows, the present calculation procedure shares many features with the parabolic one. In this paper importance is given to the differences between the two procedures; the common features are mentioned only briefly.

3.1. *Finite-difference equations*

The method of derivation of the finite-difference equations from the differential equations is identical to that in the parabolic calculation procedure [2]. The finite-difference equations are derived by integrating the differential equations over "control volumes" for individual variables transported. The three velocity components and pressure are stored in staggered positions on the finite-difference grid. The definitions of control volumes and storage of variables are shown in Fig. 1.

The difference equations can be stated as follows:

Continuity:

$$C^u\{(\rho u)_e-(\rho u)_p\}+C^v\{(\rho v)_n-(\rho v)_p\}+C^w\{(\rho w)_p-(\rho w)_u\}=0 \quad (3.1)$$

Momenta:

$$u_p = A_N^u u_n + A_S^u u_s + A_E^u u_e + A_W^u u_w + B^u + D^u(p_P - p_W) \quad (3.2)$$

$$v_p = A_N^v v_n + A_S^v v_s + A_E^v v_e + A_W^v v_w + B^v + D^v(p_P - p_S) \quad (3.3)$$

$$w_P = A_N^w w_n + A_S^w w_s + A_E^w w_e + A_W^w w_w + B^w + D^w(p_D - p_P). \quad (3.4)$$

Property, ϕ:

$$\phi_P = A_N^\phi \phi_N + A_S^\phi \phi_S + A_E^\phi \phi_E + A_W^\phi \phi_W + B^\phi. \quad (3.5)$$

In the above equations, the A coefficients express the combined effects of convection and diffusion, linking the property at P with its neighbours in the cross-stream plane (Fig. 1); the B coefficients express the contribution of upstream convection and of source terms, expressed by "S" in the differential equations. The "C"s represent areas of cell faces across which

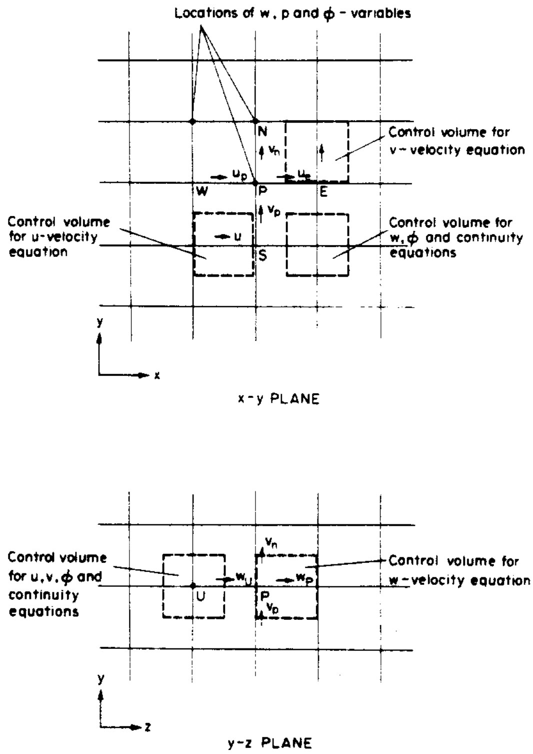

FIG. 1. Storage locations and control volumes for dependent variables.

mass is convected; and the "D"s are coefficients linking pressure differences to corresponding velocities. The subscripts P, N, S, E, W and U refer to variables at the grid nodes; and the subscripts p, n, s, e and w denote the variables at the interface locations shown in Fig. 1.

3.2. *Sequence of calculation steps*

The above difference equations are solved by an iterative procedure. All variables except the pressure are stored in two-dimensional arrays and are evaluated, over cross-stream planes, by marching in the predominant flow direction. The pressure field is stored three-dimensionally, and is first assigned a guessed value; it is then updated by sweeping repeatedly through the flow domain so as to remove errors in continuity and momentum.

The sequence of calculation steps is the following:

1. The three-dimensional pressure field is first assigned guessed values.

2. A march through the flow domain is initiated; and, from the inlet distributions of u, v and w their distributions at the next downstream location are calculated. The pressure gradient terms are evaluated from the guessed pressure field; and the coefficients A, B etc. are evaluated from variables in store at that instant. The equations are solved using a tridiagonal matrix algorithm (details are given in [2]).

3. The newly calculated distributions of u, v and w are checked for satisfaction of mass continuity at all the grid locations in the cross stream plane. The pressure and velocity fields are then corrected by solving a pressure-correction equation so as to remove

errors in mass continuity. The derivation and solution of the pressure-correction equation are described in the Appendix.

4. The equation for property ϕ is solved so as to provide distributions appropriate to the new downstream axial station.

5. Another new downstream axial station is chosen and the momentum, continuity and ϕ-equations are solved as described above. This step-wise march is continued until the end of the flow domain is reached. By the end of one complete marching sweep, a new three-dimensional distribution of pressure has been obtained.

6. Steps 2, 3, 4 and 5 are then repeated until the pressure corrections, or the continuity errors which give rise to them, have become smaller than a preassigned value. On the last sweep, the converged distribution of velocities, pressure, shear stresses, temperature etc. are printed out, as are needed.

3.3. *The boundary conditions*

The hydrodynamic boundary conditions governing the flow situation are prescribed through specified distributions of either velocities or pressure. When all the boundaries are of specified *velocity*, it is necessary, for *incompressible* flows, to fix one pressure point as a datum to the rest of the pressure field. In *compressible* partially-parabolic flows however, this is not necessary as the density level will decide the pressure level. The thermal boundary conditions are prescribed either as prescribed temperature or as prescribed heat flux at the boundaries.

4. AN APPLICATION OF THE CALCULATION PROCEDURE

This section describes an application of the calculation procedure. The physical situation considered is shown in Fig. 2: fluid flows through a square duct in which a wire screen is situated midway between inlet and outlet; the screen occupies only a portion of the cross-sectional area, and, in that region, creates a sink of axial momentum expressed by the relation

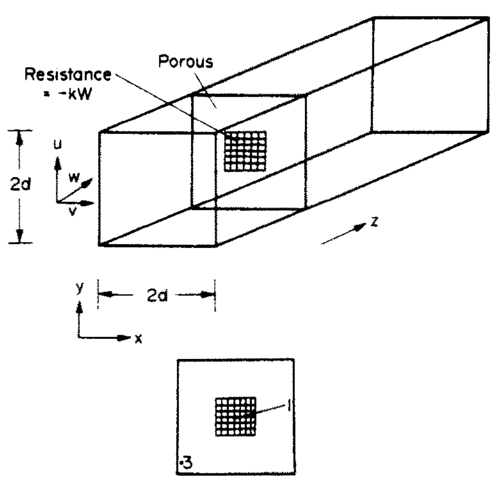

FIG. 2. Geometry considered.

$$S_w = -Kw \tag{4.1}$$

where K is a constant.

Because of the wire screen, the pressure in the centre of the duct rises to compensate for the additional pressure drop. This increase in pressure also retards the axial flow, thus diverting the streamlines away from the screen. The flow region further upstream of the screen also experiences the pressure rise and the bending of the streamlines, but to an extent diminishing with distance from the screen. Thus the flow is influenced by events downstream through the pressure field. The flow is partially-parabolic.

Calculations have been made for the above physical situation using the partially-parabolic calculation procedure. For ease of interpretation of the results, the duct walls have been considered to be frictionless, and the flow to be laminar. The finite-difference grid, for the typical calculations presented here, possessed 10 nodes in the x and y directions, and 40 grid nodes in the z-direction; the procedure, under the above conditions, converged in 18 sweeps of the flow domain.

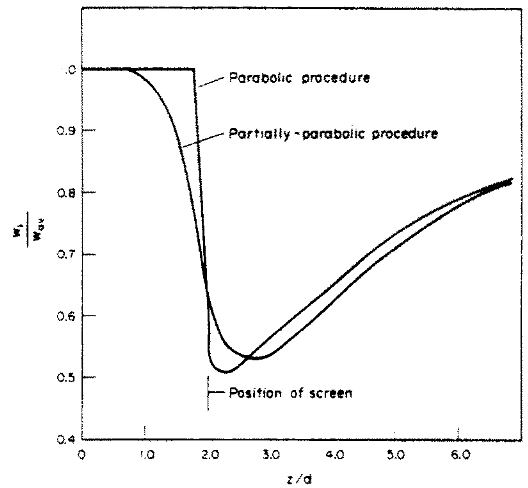

FIG. 3. Development of axial velocity at point 1; w_1 is the velocity at point 1 and w_{av} is the bulk-average velocity.

Figure 3 displays the predicted development of the centre-line axial velocity; and Fig. 4 displays the pressure variation at three cross-stream locations. Also shown are the results from a parabolic calculation, using the procedure of [2]. It is seen that the partially-parabolic calculations display the expected behaviour of the flow. The parabolic calculations show a jump in the pressure and velocity only when the screen is reached; further, as a result of the incorrect upstream flow field, the flow downstream of the screen is in error. It is therefore necessary to employ a partially-parabolic calculation scheme to predict the above flow situation.

5. CONCLUDING REMARKS

In the present paper, we have described a calculation procedure for partially-parabolic flow situations. Its benefits have been demonstrated by its application to a typical partially-parabolic flow problem.

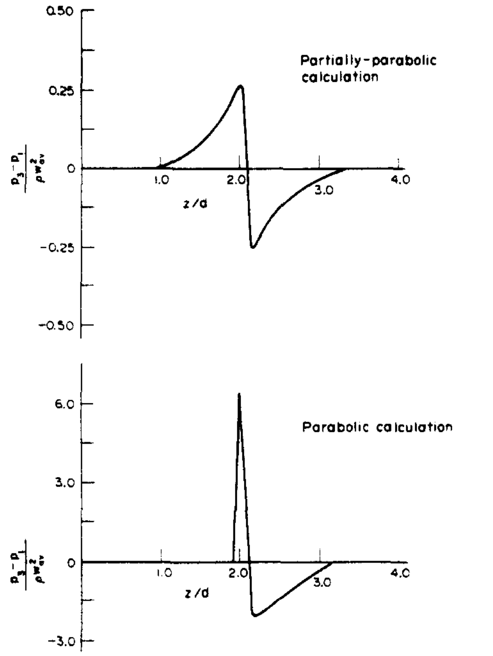

FIG. 4. Development of cross-stream pressure variation; points 1 and 3 are shown in Fig. 2; w_{av} is the bulk-average velocity. Note the difference in the vertical scales of the above two figures.

There are many practically occurring flow situations which need the present calculation procedure for their solution: a few of them are:

(a) Flow and heat transfer in a pipe bend, or in a tightly-wound spiral pipe.
(b) Flow of warm water from a power station, discharged into a river bend.
(c) Film cooling by discharge of coolant air from a row of inclined holes in the surface of a turbine blade.
(d) Flow in turbomachinery blade passages.
(e) The mixing of dilution air with combustion products in the downstream portion of a gas-turbine combustor.

Work on these is currently being conducted by the authors and their colleagues.

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REFERENCES

1. L. S. Caretto, A. D. Gosman, S. V. Patankar and D. B. Spalding, Two calculation procedures for steady three-dimensional flows with recirculation, in *Proceedings of Third International Conference on Numerical Methods in Fluid Dynamics*, Paris (July 1972).
2. S. V. Patankar and D. B. Spalding, A calculation procedure for three-dimensional parabolic flows, *Int. J. Heat Mass Transfer* **15**, 1787 (1972).
3. V. S. Pratap, Flow and heat transfer in curved ducts, Ph.D. Thesis, London University (1975).

APPENDIX

A1. *The Pressure-Correction Equation*

A1.1. *Derivation*

The purpose of the pressure-correction equation is to correct the pressure and velocity fields so that mass continuity is satisfied at all grid locations in the flow domain. The pressure-correction equation is derived from the continuity equation and simplified forms of the momentum equations. This appendix describes the derivation and solution procedure for the pressure-correction equation.

1. First, the velocity and pressure fields are expressed as:

$$\begin{aligned} p &= p^* + p', \\ u &= u^* + u' \quad \text{and} \quad w = w^* + w' \\ v &= v^* + v' \end{aligned} \tag{A1.1}$$

where the primed quantities represent corrections to the best-estimate (asterisked) values.

2. The corrections to velocities are then related to the pressure corrections, by differentiation of the finite-difference momentum equations; only the central velocity of the control volume is allowed to vary during this differentiation. Thus:

$$u'_p = D^u(p'_P - p'_W) \tag{A1.2}$$

where D^u is the coefficient in equation (3.2).

3. The expressions for velocity corrections, along with (A1.1), are substituted into the finite-difference form of the continuity equation; and the coefficients of pressure corrections are rearranged. The equation so derived is of the following form

$$A^p_P p'_P = A^p_N p'_N + A^p_E p'_E + A^p_W p'_W + A^p_S p'_S + A^p_D p'_D + A^p_U p'_U + m_P. \tag{A1.3}$$

The A coefficients involve areas and the D^u, D^v and D^w coefficients. A^p_P is given by

$$A^p_P = A^p_E + A^p_W + A^p_N + A^p_S + A^p_U + A^p_D. \tag{A1.4}$$

A1.2. *Solution of the pressure-correction equation*

Unlike the momentum equations, the pressure-correction equation has two additional terms $A^p_D p'_D$ and $A^p_U p'_U$ which link downstream and upstream pressure corrections to p'_P. Because of these links the equation is three-dimensional; i.e. a change of p' at D effects a change at U, and further upstream; so pressure-corrections need to be stored three-dimensionally. However, to avoid the three-dimensional storage of pressure-corrections, the pressure-correction equations are solved, in the present procedure, on cross-stream planes by repeated application of the tri-diagonal matrix algorithm. During the solution p'_U and p'_D are taken as equal to zero. By doing so, the pressures are updated as the marching sweep is contained.

In addition to the corrections dictated by the pressure-correction equation, the pressure field is further corrected in the following two ways. These two corrections have been observed to procure faster convergence of the numerical scheme.

1. A block adjustment (i.e. a uniform pressure increment over the plane) is applied at a plane downstream of that of P, to satisfy the overall mass-flow balance.

2. Certain fractions of the calculated pressure-corrections at any cross stream plane are applied also to pressures at upstream locations. The amounts of pressure-corrections depend upon the grid sizes, the nearness of the upstream location and the coefficients. It has been found that when this correction is made, the downstream events are transferred upstream at a faster rate. The details of the expressions used are given in [3].

ECOULEMENT TRIDIMENSIONNELS EN CONDUITE AVEC TRANSFERT THERMIQUE

Résumé—On décrit une procédure numérique de calcul applicable à des configurations tridimensionnelles d'écoulements en conduite de nature semi-parabolique, c'est à dire à des écoulements dans lesquels l'effet de la convection est dirigé longitudinalement vers l'aval, celui de la diffusion transversalement à l'écoulement, tandis que l'influence des régions en aval sur les régions en amont est transmise par le champ de pression. La procédure numérique permet de traiter économiquement de tels écoulements; la pression est mise en mémoire dans un tableau à trois dimensions tandis que les autres variables sont placées dans des tableaux à deux dimensions. A titre d'illustration, les résultats numériques obtenus dans une application de la procédure de calcul sont comparés à ceux obtenus à l'aide d'une procédure parabolique.

STRÖMUNGSVERHALTEN UND WÄRMEÜBERGANG IN DREIDIMENSIONALEN KANALSTRÖMUNGEN

Zusammenfassung—Es wird ein Berechnungsverfahren für dreidimensionale Kanalströmungen beschrieben. Es handelt sich hierbei um Verhältnisse teilweise parabolischer Natur, wobei konvektive Einflüsse sich nur stromabwärts auswirken, die Diffusion quer zur Strömung wirkt und lediglich über den Druck auch eine Beeinflussung entgegen der Strömungsrichtung auftreten kann. Die numerische Berechnungsmethode erweist sich als sehr geeignet für solche Strömungen; der Druck wird in dreidimensionaler Form eingegeben, während die anderen Variablen zweidimensional betrachtet werden. Zur Erläuterung werden für einen Anwendungsfall die Ergebnisse nach dieser Methode mit denen nach der parabolischen Berechnungsmethode verglichen.

ТЕЧЕНИЕ ЖИДКОСТИ И ТЕПЛООБМЕН В ТРЕХМЕРНОМ КАНАЛЕ

Аннотация — Рассматривается метод расчета трехмерных течений с профилем скорости, близким к параболическому, когда конвективный перенос имеет место только вниз по потоку, а диффузионный — поперек потока. Перенос из области вниз по потоку в область вверх по потоку осуществляется за счет давления. Численный расчет таких течений довольно прост. В этом случае давление является трехмерной величиной, а остальные переменные — двумерными. Для иллюстрации проведено сравнение полученных результатов с результатами расчета параболического случая.